Suggested Solutions to:
Final (Resit) Exam, August 13, 2021
Industrial Organization

This version: August 30, 2021

Question 1: Collusion with fluctuating, asymmetric, and persistent cost

(a) Derive the expression for $V_{1,L}^{n}$ stated in (7) [in the exam paper].

If the state is low, firm 1 earns a zero profit. In any given future period, firm 1 earns a zero profit if the state is low and the profit c if the state is high. As the two states occur with equal probability (regardless of the current state), the firm's expected profit in any future period equals $\frac{c}{2}$. We can thus write firm 1's present-discounted stream of profits in the punishment phase, given that the current state is low, as

$$\begin{aligned} V_{1,L}^{n} &= 0 + \delta \frac{c}{2} + \delta^2 \frac{c}{2} + \delta^3 \frac{c}{2} + \cdots \\ &= \delta \frac{c}{2} \left(1 + \delta + \delta^2 + \delta^3 \cdots\right) \\ &= \frac{\delta c}{2(1-\delta)}, \end{aligned} \tag{1}$$

where the last equality uses the standard formula for the sum of an infinite geometric series.

An alternative way of deriving the expression for $V_{1,L}^{n}$ is to note that $V_{1,L}^{n}$ and $V_{1,H}^{n}$ must satisfy the following two equations:

$$V_{1,L}^{n} = 0 + \delta \left[\frac{1}{2} V_{1,L}^{n} + \frac{1}{2} V_{1,H}^{n}\right], \qquad V_{1,H}^{n} = c + \delta \left[\frac{1}{2} V_{1,L}^{n} + \frac{1}{2} V_{1,H}^{n}\right]. \tag{2}$$

That is, $V_{1,L}^{n}$ can be written as 0 (the profit in the current period if the state is low) plus δ (the discount factor) multiplied by the expected stream of profits from the next period and onward. That expected stream of profits can be written in terms of the same variables $V_{1,L}^{n}$ and $V_{1,H}^{n}$ that show up on the left-hand sides of (2), as the horizon is infinite. The term $\frac{1}{2}$ is the probability that a low (or a high) state occurs in the next period. The equation for $V_{1,H}^{n}$ can be understood in a similar way (but here the profit in the current period equals c).

Once we have the two equations in (2), it is straightforward to solve them for $V_{1,L}^{n}$ and $V_{1,H}^{n}$ (as the equations are linear in these variables). Doing that yields

$$V_{1,L}^{n} = \frac{\delta c}{2(1-\delta)}, \qquad V_{1,H}^{n} = \frac{(2-\delta)c}{2(1-\delta)}. \tag{3}$$

(b) Derive the conditions stated in (10) and (11) [in the exam paper], i.e., the conditions $\alpha_L \leq \psi_2^L(\alpha_H)$ and $\alpha_L \leq \psi_2^H(\alpha_H)$. As the notation indicates, these conditions refer to the incentives of firm 2. To "derive the conditions" here means to show rigorously that firm 2 does not have an incentive to deviate unilaterally from the trigger strategy if, and only if, (10) and (11) hold.

Firm 2 does not have an incentive to deviate unilaterally from the trigger strategy if, and only if, the following four conditions are satisfied:

(i) Firm 2 must not want to deviate **on the equilibrium path** and in a period when the **low** state has realized.

(ii) Firm 2 must not want to deviate **on the equilibrium path** and in a period when the **high** state has realized.

(iii) Firm 2 must not want to deviate **off the equilibrium path** and in a period when the **low** state has realized.

(iv) Firm 2 must not want to deviate **off the equilibrium path** and in a period when the **high** state has realized.

We can note that the conditions under (iii) and (iv)—the ones that concern the incentives to deviate off the equilibrium path—are both satisfied, as the trigger strategy in those situations prescribes behavior that is part of a one-shot Nash equilibrium.[1]

- Consider condition (i) for firm 2. This can be written as

$$V_2^L \geq 1 - \varepsilon + \delta \times 0 \tag{4}$$

for all $\varepsilon > 0$. The left-hand side of the above inequality is, by definition, firm 2's expected stream of profits if not deviating, given that the current state is low. The first term on the right-hand side, $1 - \varepsilon$, is the firm's (single-period) profit if undercutting the rival slightly, thereby getting all the demand. The term ε can be made arbitrarily small, as firm 2 can undercut with a price that is arbitrarily close to the collusive price (which equals one) but still strictly lower. The second term on the right-hand side is firm 2's expected stream of profits when being in the punishment phase (which equals zero), discounted with the discount factor δ (although this does not matter as it is multiplied by zero). The condition in (4) holds for all $\varepsilon > 0$ if, and only if, it holds for $\varepsilon = 0$. That is, the condition can equivalently be written as $V_2^L \geq 1$. This inequality can in turn, by using the expression for V_2^L stated in the question, be written as

$$\frac{(2-\delta)(1-\alpha_L) + \delta(1-\alpha_H)}{2(1-\delta)} \geq 1 \Leftrightarrow \alpha_L \leq \frac{\delta(2-\alpha_H)}{2-\delta} \stackrel{\text{def}}{=} \psi_2^L(\alpha_H). \tag{5}$$

- Next, consider condition (ii) for firm 2. This condition can be written as:

$$V_2^H \geq 1 - c - \varepsilon + \delta \times 0 \tag{6}$$

for all $\varepsilon > 0$, where the interpretation is similar to the one for (4). Also the condition in (6) holds for all $\varepsilon > 0$ if, and only if, it holds for $\varepsilon = 0$, which means that it is equivalent to $V_2^H \geq 1 - c$. This inequality can in turn, by using the expression for V_2^H stated in the question, be written as

$$\frac{\delta(1-\alpha_L) + (2-\delta)(1-\alpha_H)}{2(1-\delta)} \geq 1 - c \Leftrightarrow \alpha_L \leq \frac{2\delta + 2(1-\delta)c - (2-\delta)\alpha_H}{\delta} \stackrel{\text{def}}{=} \psi_2^H(\alpha_H). \tag{7}$$

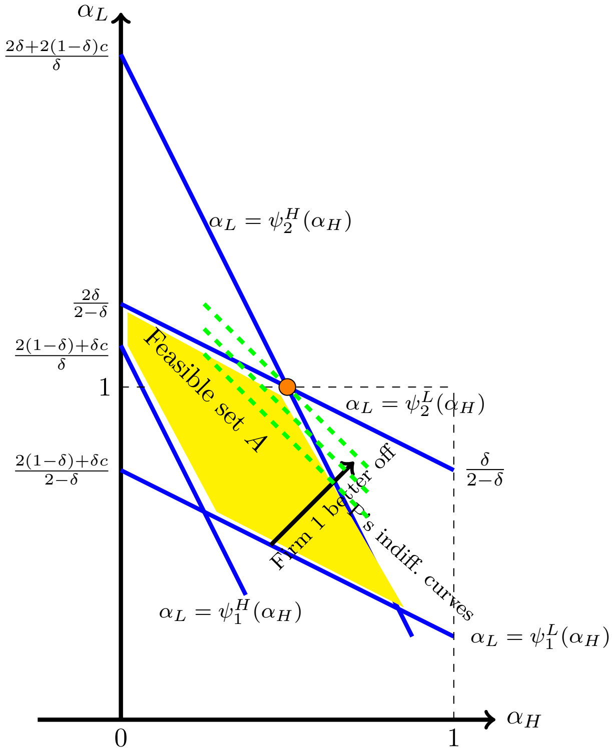
It follows from the above reasoning that firm 2 does not have an incentive to deviate unilaterally from the trigger strategy if, and only if, (5) and (7)—which are identical to (10) and (11) in the question—hold.

(c) What values of α_L and α_H maximize firm 1's expected stream of profits, $\frac{1}{2}\left(V_1^L + V_1^H\right)$, subject to the constraint $(\alpha_L, \alpha_H) \in A$?

To solve the problem, it is easiest to use a graphical analysis. Consider the figure below, which depicts the (α_H, α_L)-space. The feasible set, A, is defined as the area above or on the graphs of $\alpha_L = \psi_1^L(\alpha_H)$ and $\alpha_L = \psi_1^H(\alpha_H)$, below or on the graphs of $\alpha_L = \psi_2^L(\alpha_H)$ and $\alpha_L = \psi_2^H(\alpha_H)$, and satisfying $\alpha_L \in [0,1]$ and $\alpha_H \in [0,1]$. This area is colored yellow in the figure. The feasible set A must, in qualitative terms, look as in figure, because we can verify the following key relationships. First, the vertical intercept of $\psi_1^L(\alpha_H)$ always lower than that of $\psi_2^L(\alpha_H)$:

$$\frac{2(1-\delta) + \delta c}{2-\delta} < \frac{2\delta}{2-\delta} \Leftrightarrow c < \frac{2(2\delta - 1)}{\delta}, \tag{8}$$

[1] The argument for why that behavior is a Nash equilibrium, given the tie-breaking rule that is assumed in the question, is standard and was discussed in the course (as our textbook assumes this tie-breaking rule).



which always holds as the last right-hand side is weakly larger than one (given the assumption that $\delta \geq \frac{2}{3}$). Second, the vertical intercept of $\psi_1^H(\alpha_H)$ is always lower than that of $\psi_2^H(\alpha_H)$:

$$\frac{2(1-\delta)+\delta c}{\delta} < \frac{2\delta+2(1-\delta)c}{\delta} \Leftrightarrow c < \frac{2(2\delta-1)}{3\delta-2}, \tag{9}$$

which always holds as the last right-hand side is larger than one. Third and finally, the graph of $\psi_2^H(\alpha_H)$ crosses the graph of $\psi_2^L(\alpha_H)$ for some $\alpha_H < 1$:

$$\psi_2^H(1) = \frac{2\delta+2(1-\delta)c-(2-\delta)\times 1}{\delta} < \frac{\delta}{2-\delta} \Leftrightarrow c < \frac{2(1-\delta)}{2-\delta}, \tag{10}$$

which holds by assumption.

Firm 1's objective function is given by

$$\frac{1}{2}\left(V_1^L + V_1^H\right) = \frac{1}{2}\left(\alpha_L + \alpha_H\right)$$

The associated iso-profit curves are thus straight lines with slope -1. A few such curves are shown in the figure (in green). Because the graph of $\psi_2^L(\alpha_H)$ has a flatter slope ($-\frac{\delta}{2-\delta} > -1$) and the graph of $\psi_2^H(\alpha_H)$ has a steeper slope ($-\frac{\delta}{2-\delta} < -1$), it is geometrically obvious that the optimum must be located at the crossing of the two straight lies $\alpha_L = \psi_2^L(\alpha_H)$ and $\alpha_L = \psi_2^H(\alpha_H)$, so at the orange dot in the figure.

Conclusion: The unique optimum is at the point where firm 2's two constraint are both binding or, if we solve for that point, at $(\alpha_H, \alpha_L) = \left(\delta + \frac{2-\delta}{2}c, \delta - \frac{\delta}{2}c\right)$.

Question 2: Vertically related firms and RPM

Part (a)

We can solve for the subgame-perfect equilibrium with the help of backward induction. We thus begin by solving the downstream firm's problem. From the question, the downstream firm's profit equals

$$\pi^D = (1-p)\,(p-w)\,\frac{e}{a+e} - e.$$

The first-order condition w.r.t. p can be written as

$$\frac{\partial \pi^D}{\partial p} = \left[-\left(p-w\right)+\left(1-p\right)\right]\frac{e}{a+e} = 0 \Rightarrow \widehat{p} = \frac{1+w}{2}. \tag{11}$$

The first-order condition w.r.t. e is given by

$$\frac{\partial \pi^D}{\partial e} = \left(1-p\right)\left(p-w\right)\frac{a}{\left(a+e\right)^2} - 1 = 0$$

or, equivalently (using (11)),

$$\left(a+\widehat{e}\right)^2 = a\left(1-\widehat{p}\right)\widehat{p}-w) = a\left(\frac{1-w}{2}\right)^2 \Rightarrow \widehat{e} = \sqrt{a}\left(\frac{1-w}{2}\right) - a. \tag{12}$$

In the last step above, we can safely ignore the negative root as we know that $e \geq 0$. For later use, note that

$$\frac{\widehat{e}}{a+\widehat{e}} = \frac{\sqrt{a}\left(\frac{1-w}{2}\right)-a}{\sqrt{a}\left(\frac{1-w}{2}\right)} = 1 - \frac{2\sqrt{a}}{1-w}. \tag{13}$$

Next, consider the first stage, where the upstream firm chooses w. From the question, the upstream firm's profit is

$$\begin{aligned}
\pi^U &= \left(1-\widehat{p}\right)\frac{\widehat{e}}{a+\widehat{e}}w \\
&= \left(\frac{1-w}{2}\right)\left[1-\frac{2\sqrt{a}}{1-w}\right]w \\
&= \frac{w\left(1-2\sqrt{a}-w\right)}{2},
\end{aligned}$$

where the second line uses (11) and (13).

The upstream firm's first-order condition can be written as

$$\frac{\partial \pi^U}{\partial w} = \frac{1-2\sqrt{a}-2w}{2} = 0 \Rightarrow w^* = \frac{1-2\sqrt{a}}{2}. \tag{14}$$

By the assumption $a < \frac{1}{4}$, this expression for w^* is strictly positive. This means that the optimal wholesale price is not at a corner solution, which we implicitly assumed when formulating the first-order condition with an equality. Plugging (14) back into (11) and (12), we obtain

$$p^* = \frac{1+w^*}{2} = \frac{1}{2}\left[1+\frac{1-2\sqrt{a}}{2}\right] = \frac{3-2\sqrt{a}}{4}$$

and

$$\begin{aligned}
e^* &= \sqrt{a}\left(\frac{1-w^*}{2}\right) - a \\
&= \frac{\sqrt{a}}{2}\left(1-\frac{1-2\sqrt{a}}{2}\right) - a \\
&= \frac{\sqrt{a}}{4}\left(1+2\sqrt{a}\right) - a \\
&= \frac{\sqrt{a}\left(1-2\sqrt{a}\right)}{4}.
\end{aligned}$$

We can again note that the derived expressions are both positive, thanks to the assumption that $a < \frac{1}{4}$.

Summing up, we have that the equilibrium values of p and e are given by

$$p^* = \frac{3-2\sqrt{a}}{4}, \qquad e^* = \frac{\sqrt{a}\left(1-2\sqrt{a}\right)}{4}.$$

Part (b)

A simple comparison tells us that $p^* > p^I$ and $e^* < e^I$. The logic behind these relationships is that there is a positive externality between the firms, which is not taken into account when the downstream firm is a separate firm. In particular, both a lower consumer price and a larger effort level increase trade, which has a positive impact (also) on the upstream firm's profit. Therefore, when the firms are integrated, they will choose a lower price and a higher effort.

We should expect consumer surplus (in expected terms) to be larger if the price is lower (for then demand is larger, given a high demand realization) and if the effort is higher (for then the probability of a high demand state is larger). We saw that we indeed have both $p^I < p^*$ and $e^I > e^*$. Therefore, integration should yield the largest (expected) consumer surplus.

Part (c)

Again, we can solve for the subgame perfect equilibrium by using backward induction, beginning with the downstream firm's problem. From the question, the downstream firm's profit equals

$$\pi^D = (1-p)(p-w)\frac{e}{a+e} - e.$$

The first-order condition w.r.t. e is given by

$$\frac{\partial \pi^D}{\partial e} = (1-p)(p-w)\frac{a}{(a+e)^2} - 1 = 0$$

or, equivalently,

$$(a+\widehat{e})^2 = a(1-p)(p-w) \Rightarrow \widehat{e} = \sqrt{a(1-p)(p-w)} - a. \tag{15}$$

As in part (a), the negative root is not relevant as we know that $e \geq 0$. For later use, note that

$$\frac{\widehat{e}}{a+\widehat{e}} = \frac{\sqrt{a(1-p)(p-w)} - a}{\sqrt{a(1-p)(p-w)}} = 1 - \frac{\sqrt{a}}{\sqrt{(1-p)(p-w)}}. \tag{16}$$

Next, consider the first stage, where the upstream firm chooses w and p. From the question, the upstream firm's profit is

$$\begin{aligned}
\pi^U &= (1-p)\frac{\widehat{e}}{a+\widehat{e}}w \\
&= (1-p)\left[1 - \frac{\sqrt{a}}{\sqrt{(1-p)(p-w)}}\right]w \\
&= (1-p)w - \sqrt{a}\frac{\sqrt{1-p}}{\sqrt{p-w}}w \\
&= (1-p)w - \sqrt{a}(1-p)^{\frac{1}{2}}(p-w)^{-\frac{1}{2}}w,
\end{aligned}$$

where the second line uses (16). The upstream firm's first-order condition w.r.t. w can be written as

$$\frac{\partial \pi^U}{\partial w} = (1-p) - \sqrt{a}(1-p)^{\frac{1}{2}}\left[\frac{1}{2}(p-w)^{-\frac{3}{2}}w + (p-w)^{-\frac{1}{2}}\right] = 0 \Leftrightarrow$$

$$(1-p)^{\frac{1}{2}} = \sqrt{a}(p-w)^{-\frac{3}{2}}\left[\frac{1}{2}w + (p-w)\right] \Leftrightarrow 2(1-p)^{\frac{1}{2}}(p-w)^{\frac{3}{2}} = \sqrt{a}(2p-w). \tag{17}$$

Similarly, the first-order condition w.r.t. p can be written as

$$\frac{\partial \pi^U}{\partial p} = -w - \sqrt{a}w\left[-\frac{1}{2}(1-p)^{-\frac{1}{2}}(p-w)^{-\frac{1}{2}} - \frac{1}{2}(1-p)^{\frac{1}{2}}(p-w)^{-\frac{3}{2}}\right] = 0$$

$$\Leftrightarrow 1 = \frac{\sqrt{a}}{2}(1-p)^{-\frac{1}{2}}(p-w)^{-\frac{3}{2}}\left[(p-w)+(1-p)\right] \Leftrightarrow 2(1-p)^{\frac{1}{2}}(p-w)^{\frac{3}{2}} = \sqrt{a}(1-w). \tag{18}$$

Combining (17) and (18), we have

$$\sqrt{a}(2p-w) = \sqrt{a}(1-w) \Rightarrow p^R = \frac{1}{2}.$$

The equilibrium price in this model with resale price maintenance is therefore the same as the price under integration, which was stated in the question.

In summary,

$$p^R = \frac{1}{2}, \qquad p^R = p^I$$

In order to answer the last part of the question, plug $p^R = \frac{1}{2}$ into (15) to obtain

$$\begin{aligned} e^R &= \sqrt{a\left(1-p^R\right)\left(p^R-w^R\right)} - a \\ &= \frac{1}{2}\sqrt{a\left(1-2w^R\right)} - a \\ &= \frac{\sqrt{a}\left(\sqrt{1-2w^R}-\sqrt{a}\right)}{2}. \end{aligned}$$

The equilibrium value of w^R is implicitly defined by (17), evaluated at $w = w^R$ and $p = p^R$:

$$2\left(1-p^R\right)^{\frac{1}{2}}\left(p^R-w^R\right)^{\frac{3}{2}} = \sqrt{a}\left(2p-w^R\right) \Leftrightarrow 2\left(\frac{1}{2}\right)^{\frac{1}{2}}\left(\frac{1}{2}-w^R\right)^{\frac{3}{2}} = \sqrt{a}\left(1-w^R\right) \Leftrightarrow$$

$$\left(1-2w^R\right)^{\frac{3}{2}} = 2\sqrt{a}\left(1-w^R\right) \Leftrightarrow \varphi\left(\widehat{w}\right) = 0, \quad \text{where } \varphi\left(w\right) \stackrel{\text{def}}{=} \left(1-2w\right)^{\frac{3}{2}} - 2\sqrt{a}\left(1-w\right).$$

The function $\varphi(w)$ satisfies $\varphi(0) > 0$, $\varphi\left(\frac{1}{2}\right) < 0$, $\varphi'(0) < 0$, and $\varphi'\left(\frac{1}{2}\right) > 0$. Moreover, it is convex on the interval $[0, \frac{1}{2}]$. (You may want to draw a figure to illustrate this.) It follows that w^R is the unique value of w where the graph of $\varphi(w)$ crosses the horizontal axis in the figure from above.

The last part of the question concerns the relationship between e^R and e^I. We have

$$\begin{aligned} e^R < e^I &\Leftrightarrow \frac{\sqrt{a}\left(\sqrt{1-2w^R}-\sqrt{a}\right)}{2} < \frac{\sqrt{a}\left(1-2\sqrt{a}\right)}{2} \Leftrightarrow \sqrt{1-2w^R} < 1-\sqrt{a} \\ &\Leftrightarrow w^R > \frac{1-\left(1-\sqrt{a}\right)^2}{2} = \frac{\sqrt{a}\left(2-\sqrt{a}\right)}{2}. \end{aligned} \tag{19}$$

By using the characterization of w^R above and referring to the figure, we obtain the result that $e^R < e^I$ if and only if the graph of $\varphi(w)$, evaluated at the cutoff in (19), lies above the horizontal axis. That is,

$$\begin{aligned} e^R < e^I &\Leftrightarrow \varphi\left(\frac{\sqrt{a}\left(2-\sqrt{a}\right)}{2}\right) > 0 \Leftrightarrow \left(1-2\frac{\sqrt{a}\left(2-\sqrt{a}\right)}{2}\right)^{\frac{3}{2}} > 2\sqrt{a}\left(1-\frac{\sqrt{a}\left(2-\sqrt{a}\right)}{2}\right) \\ &\Leftrightarrow \left(1-\sqrt{a}\right)^3 > \sqrt{a}\left(2-2\sqrt{a}+a\right). \end{aligned} \tag{20}$$

However, it is easy to see that the inequality in (20) is satisfied for $a = 0$, but it is violated (indeed, holds with the opposite inequality) for $a = 1/4$. It follows that it is alternative (iv) in the question that is true:

Whether e^R is smaller or larger than e^I depends on the value of a.

Appendix

Here I derive the conditions $\alpha_L \geq \psi_1^L(\alpha_H)$ and $\alpha_L \geq \psi_1^H(\alpha_H)$ that are stated in the exam question (although providing these derivations are not needed to answer the questions in the exam).

- Consider condition (i) for firm 1. This condition can be written as:

$$V_1^L \geq 1 - \varepsilon + \delta\left[\frac{1}{2}V_{1,L}^n + \frac{1}{2}V_{1,H}^n\right] = 1 - \varepsilon + V_{1,L}^n \tag{21}$$

  for all $\varepsilon > 0$, where the equality holds thanks to (2). The condition in (21) holds for all $\varepsilon > 0$ if, and only if, it holds for $\varepsilon = 0$. That is, the condition can equivalently be written as $V_1^L \geq 1 + V_{1,L}^n$. This inequality can in turn, by using the expressions for V_1^L and $V_{1,L}^n$ stated in the question, be written as

$$\frac{(2-\delta)\alpha_L + \delta\alpha_H}{2(1-\delta)} \geq 1 + \frac{\delta c}{2(1-\delta)} \Leftrightarrow \alpha_L \geq \frac{2(1-\delta)+\delta c - \delta\alpha_H}{2-\delta} \stackrel{\text{def}}{=} \psi_1^L(\alpha_H). \tag{22}$$

- Next consider condition (ii) for firm 1. This condition can be written as:

$$V_1^H \geq 1 - \varepsilon + \delta \left[\frac{1}{2} V_{1,L}^n + \frac{1}{2} V_{1,H}^n \right] = 1 - \varepsilon + V_{1,L}^n \tag{23}$$

for all $\varepsilon > 0$, where again the equality holds thanks to (2). That is, relative to the previous case, the equilibrium stream of profits (the left-hand side) is different, but the deviation profits (the right-hand side) are the same. The inequality in (23) is equivalent to $V_1^H \geq 1 + V_{1,L}^n$, which in turn (using the expressions for V_1^H and $V_{1,L}^n$ in the question), can be rewritten as

$$\frac{\delta \alpha_L + (2-\delta)\alpha_H}{2(1-\delta)} \geq 1 + \frac{\delta c}{2(1-\delta)} \Leftrightarrow \alpha_L \geq \frac{2(1-\delta) + \delta c - (2-\delta)\alpha_H}{\delta} \stackrel{\text{def}}{=} \psi_1^H(\alpha_H). \tag{24}$$